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A MATHEMATICAL MODEL OF THE ULTRASONIC DETECTION OF THREE-DIMENSIONAL CRACKS†

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A mathematical model of the main stages of the process of ultrasonic detection of deep three-dimensional cracks of arbitrary orientation in a uniform medium and of horizontal interfacial cracks is developed using a single boundary-integral approach. A description of the construction of the asymptotic form of the source field and of the calculation of both the energy scattering coefficient and the generalized Auld electromechanical coupling coefficient is given. Numerical results are presented which show how the scan-image of the crack depends on the ratio of its dimensions to the wavelength, on the form and orientation in space, and also the effect of the ratio of the elastic properties of the materials on the scattering coefficient of an interfacial crack situated between them. © 2002 Elsevier Science Ltd. All rights reserved.

The interpretation of the data of ultrasonic non-destructive testing is traditionally based on ray methods of general diffraction theory [1]. In view of the asymptotic nature of the ray approach, it is only used in the high-frequency band, when the wavelength of the probing signal is much less than the characteristic dimensions of the defect. On the other hand, if the dimensions of the defect are comparable with or less than the wavelength, the use of reliable mathematical models becomes particularly important, since the reflection field in this case gives a very blurred image due to diffraction, which requires special processing to establish the size and shape of the defect. In mathematical modelling, the integral approach is usually used here instead of the ray approach, in which the problem of diffraction is reduced to boundary integral equations in the unknown jump in the displacements on the crack edges [2–5]. The reflection field in this case is described by an accurate integral representation, from which one can obtain the asymptotic form in the far zone, that is similar to the asymptotic form given by the ray method. In this sense the integral approach is more general – it can be used over the whole frequency band. Although the solution of the integral equation also gives rise to certain difficulties at high frequencies, they can be completely overcome [5].

On the whole, however, solving the boundary integral equations involves considerably greater computational costs than the construction of ray asymptotic forms. The computational costs increase particularly if cracks that are not of classical shape (a circle or a strip) but have a shape that is arbitrary in plan are considered, which requires the use of a grid approximation and the evaluation of multiple integrals with hypersingular kernels on discretization [2]. The use of axisymmetric basis functions enables these problems to be surmounted and reduces computational costs to a level that is acceptable when the method is employed on a personal computer (a detailed description with a brief review of relevant papers can be found in [4, 8]). This provides a basis for developing an effective mathematical model, including the main stages of ultrasonic monitoring: 1) a description of the wave field of the source $\mathbf{u}_0$, 2) the solution of the problem of diffraction by the crack, and 3) recording of the reflected signal $\mathbf{u}_1^+$ and the construction of an image of the defect, obtained by scanning.

The structure of the model is the same as in [3], but here we use other mathematical methods. In particular, the variational-difference method [4] enables us to consider not only circular cracks, as in [3], but also cracks of arbitrary shape in plan.

Situations are encountered in practice when the crack lies between materials of different moduluses. This may be a peeling region in welded, glued or metal-ceramics joints, in multilayered composite materials, etc. The presence of interfaces gives rise to additional reflected, refracted and interfacial or channelled waves, which make it difficult to distinguish the signal reflected from the defect on their background. These difficulties are expressed mathematically as a complication of the structure of the waves incident on the crack and of the kernel of the boundary integral equations.

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1. A GENERAL DESCRIPTION OF THE MODEL

The sample under test is modelled by an elastic isotropic half-space which occupies a volume $-\infty \leqslant x_1$, $y_1 \leqslant \infty, -\infty, \leqslant z_1 \leqslant 0$ in a Cartesian system of coordinates $\mathbf{x}_1 = \{x_1, y_1, z_1\}$. The medium can be uniform, containing an arbitrarily oriented crack, occupying the plane region Ω, or double-layer (or even multilayer) with an interfacial crack in the plane $z_1 = -h$, where the layers are glued together (Fig. 1).

We will assume that the crack is situated fairly far from the outer surface of the sample, so that the field reflected for a second time from the surface can be neglected. In the case of surface cracks of cracks which reach the surface, allowance for signal rereflection between the crack and the surface leads to an integral equation of more complex form.

In this model, the medium is assumed to be isotropic and infinitely extended in a horizontal direction. The necessary physical and mathematical correctness is ensured by using the principle of limiting absorption, which is equivalent to the requirement of energy radiation at infinity [6].

On the outer surface $z_1 = 0$ there is a source of harmonic oscillations (an ultrasonic transducer), the action of which is modelled by a specified surface load $\mathbf{q}_0$

$$\boldsymbol{\tau}(\mathbf{x}_1)e^{-i\omega t}\big|_{z_1=0} = \begin{cases} \mathbf{q}_0(x_1, y_1)e^{-i\omega t}, & (x_1, y_1) \in D \\ 0, & (x_1, y_1) \notin D \end{cases} \tag{1.1}$$

Here $\boldsymbol{\tau} = \{\tau_{x_1z_1}, \tau_{y_1z_1}, \sigma_{z_1}\}$ is the surface stress vector and D is the contact area between the ultrasonic source and medium. The contact area can be arbitrary (and also disconnected: $D = \cup_k D_k$ for a system of sources). The type of source (directional or non-directional, of longitudinal or transverse waves, etc.) is defined by the form of the function $\mathbf{q}_0$.

A specified load excites an initial wave field $\mathbf{u}_0(\mathbf{x}_1)$ in the elastic medium. Diffraction of this field by the crack gives rise to a field of reflected waves $\mathbf{u}_1(\mathbf{x}_1)$ (the harmonic factor is omitted here and henceforth).

The crack is modelled by a cut of infinitesimal width with stress-free sides

$$(\boldsymbol{\tau}_0 + \boldsymbol{\tau}_1)\big|_{z=0} = 0, \quad (x, y) \in \Omega \tag{1.2}$$

Here $\mathbf{x} = \{x, y, z\}$ is a local system of coordinates, connected with the crack (the Oz axis coincides with the normal to it), $\boldsymbol{\tau}_n = T_z\mathbf{u}_n$ ($n = 0, 1$) and T_z is the stress operator for an area with normal Oz.

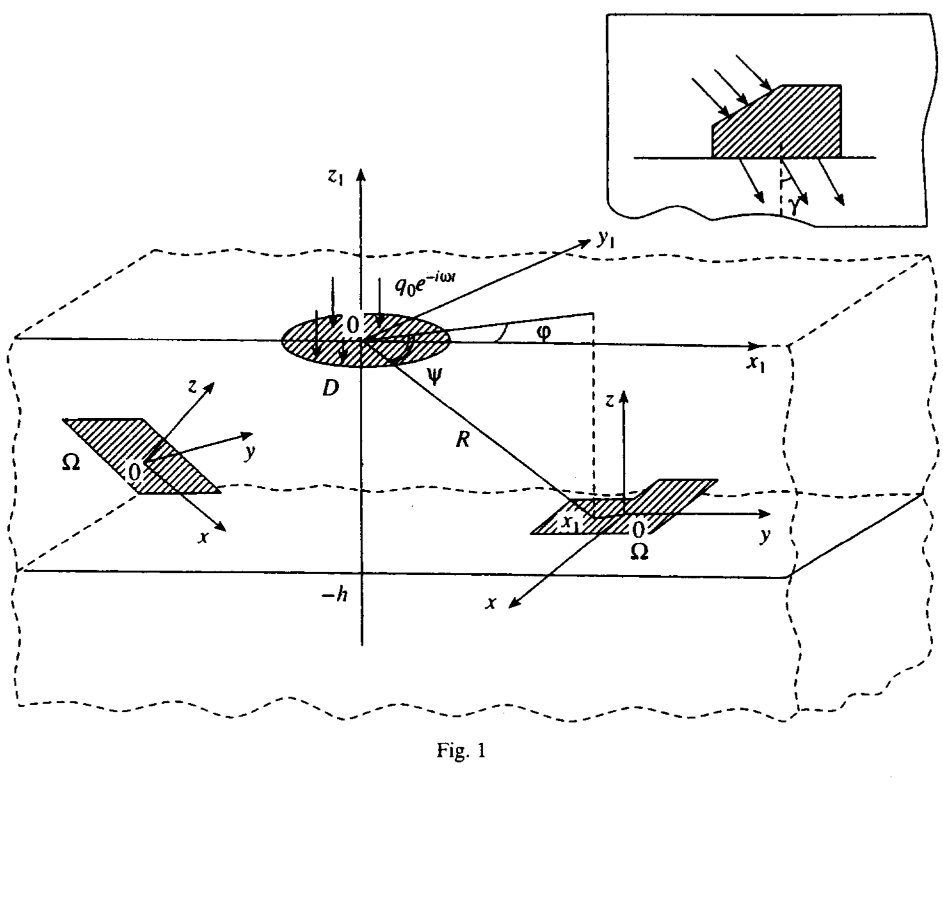

Fig. 1

The field $\mathbf{u}_0$ is continuous over the whole volume, while the reflected field is discontinuous on the cut Ω with a jump

$$\mathbf{v}(x, y) = \mathbf{u}_1(\mathbf{x})\big|_{z=0^+} - \mathbf{u}_1(\mathbf{x})\big|_{z=0^-} \neq 0, \ (x, y) \in \Omega \tag{1.3}$$

Using Green's matrix for the half-spaces $z \geqslant 0$ and $z \leqslant 0$, the reflected field $\mathbf{u}_1^{\pm}$ can be expressed in the form of oscillating contour integrals containing the unknown jump $\mathbf{v}$, which is determined from the system of integral equations which is obtained when condition (1.2) is satisfied [4]

$$\mathscr{L}\mathbf{v} \equiv \iint_{\Omega} l(x-\xi, \ y-\eta)\mathbf{v}(\xi, \ \eta)d\xi d\eta = \mathbf{f}(x, \ y), \ (x, \ y) \in \Omega \tag{1.4}$$

$$\mathbf{f}(x, \ y) = -\boldsymbol{\tau}_0\big|_{z=0}$$

An asymptotic analysis of the integral representations for $\mathbf{u}_1^{\pm}$, as before [4], gives fairly simple formulae, which enable us to construct the radiation pattern of the reflected field and obtain the values of its amplitude at required points where the signal is recorded.

Moreover, using the solution of integral equation (1.4) we can calculate the generalized Auld reflection coefficient $\delta\Gamma$ [7] (or the argument of the electromechanical dependence according to [3]), which describes the change in the signal recorded by the piezoelectric receiver, caused by the presence of the defect. For an arbitrary internal defect, surrounded by a surface S,

$$\delta\Gamma = -\frac{i\omega}{4P}\iint_S (\mathbf{u}_2 \cdot \boldsymbol{\tau}_1 - \mathbf{u}_1 \cdot \boldsymbol{\tau}_2)dS \quad \left(\mathbf{u} \cdot \boldsymbol{\tau} = \sum_j u_j \tau_j\right) \tag{1.5}$$

where $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are the stresses on the surface S corresponding to the fields $\mathbf{u}_1$ and $\mathbf{u}_2$, where we take as the first the field excited by a certain piezoelectric source in the body without a defect, while $\mathbf{u}_2$ is the field of the other source when the defect is present, and P is a certain quantity, proportional to the square of the amplitude of the electric signal of the source. Note that the dot in (1.5) denotes the sum of the products of the vector components without complex conjugation, i.e. it is not the scalar product of vectors.

Representation (1.5) also holds when the positions of the sources coincide. In this case we take as the first the initial field of the source $\mathbf{u}_0$, and as the second we take $\mathbf{u}_0 + \mathbf{u}_1$. In particular, for a crack with stress-free edges, when conditions (1.2) and (1.3) are satisfied, representation (1.5) reduces to the form

$$\delta\Gamma = -\frac{i\omega}{4P}\iint_{\Omega} \mathbf{v} \cdot \boldsymbol{\tau}_0 d\Omega \approx \frac{i\omega}{4P}\sum_{k=1}^{N} \mathbf{c}_k \cdot \mathbf{f}_k \tag{1.6}$$

The approximation of the integral of the finite sum follows in a natural way from the expansion of the unknown jump $\mathbf{v}$ in basis functions φ_k, specified at the nodes (x_k, y_k) of the grid corresponding to the variational-difference method of solving the system of integral equations (1.4) [4]. In this case $c_k = \mathbf{v}(x_k, y_k)$ is the value of the jump at nodes, found from the linear algebraic system to which Eqs (1.4) reduce on discretization (see (1.6) in [4]), while $\mathbf{f}_k = \mathbf{f}(x_k, y_k)$ form the right-hand side of this system.

The coefficient $\delta\Gamma$ is the ideal means of modelling, since it describes a quantity that can be directly measured. In the pulse-echo method, when the reflected signal is recorded at the point of radiation, the coefficient $\delta\Gamma$, for fixed parameters of the model, depends on the coordinate (x_0, y_0) of the position of the source-receiver on the surface $z_1 = 0$. Here the function $\delta\Gamma(x_0, y_0)$ gives the same image as is obtained when scanning with an ultrasonic transducer over the surface of the material being probed. One thereby eliminates the need to construct the radiation pattern of the scattered field and to trace the signal from the defect to the receiver.

Details of the realization of this model and also an analysis of the scan images obtained using it for rectangular and L-shaped cracks depending on their shape, spatial orientation, the directivity of the source and the frequency are given in an unpublished paper.†

As an example we show in Fig. 2 the form of $\delta\Gamma(x_0, y_0)$ for a horizontal L-shaped crack when scanning with a normally incident wave from a circular probe of radius 5 mm at frequencies $f = 0.5$, 1, 1.5 and

†YEKHLAKOV, A. B., The use of the generalized reflection coefficient to determine the dimensions and shape of three-dimensional cracks. Deposited at the All-Union Institute for Scientific and Technical Information, No. 3338–V00, 29 December 2000.

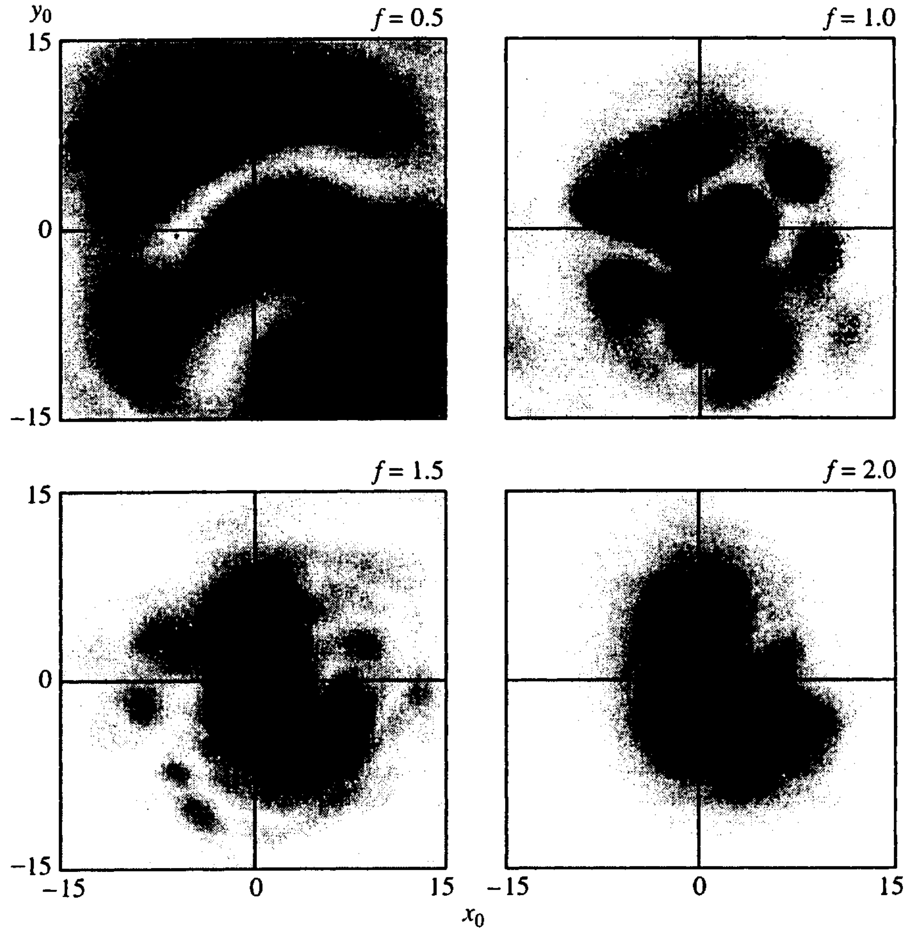

Fig. 2

2 MHz ($\delta\Gamma$ is in decibels and x_0 and y_0 are in millimetres). In order to compare the results, the parameters here and henceforth are the same as before [3], with the exception of the crack shape. The crack area is taken to be the same as for a circular crack of unit radius; the crack depth is 30mm and the material is steel with v_p=5940 m/s and v_s = 3230 m/s.

This example illustrates how the clarity of the scanned image depends on the ratio of the wavelength and the dimensions of the object. It can be seen that the image obtained at a frequency of 0.5 MHz does not enable one to determine exactly the dimensions and shape of the crack without special processing, equivalent to solving the inverse problem. Clear outlines only appear at a frequency of 2 MHz, when the ray method can start to be used. The ray method cannot be used to construct the objective function of the inverse problem at lower frequencies.

It is interesting to note that, in view of diffraction effects when the frequency changes, the blurred low-frequency image seems to rotate around the axis perpendicular to the plane of the figure. This rotation can be seen more clearly in the case of an oblong rectangular crack, which is presented in the paper cited in the footnote.

2. THE SOURCE FIELD

When solving the diffraction problem it is usually assumed that a plane P- or S-wave is incident on the crack at a certain angle to the normal $\mathbf{n}$. This enables the dependence of the scattering coefficient Σ and the radiation pattern on the angle of incidence and the frequency to be investigated, but to calculate $\delta\Gamma(x_0, y_0)$ the incident field must be related to the source.

In the case of a non-interfacial crack the wave field excited in a uniform elastic half-space of a surface load $\mathbf{q}_0$, as is well known, can be expressed in integral form in terms of Green's matrix of the half-space

k and its Fourier representation K.

$$\mathbf{u}_0(\mathbf{x}_1)=\iint\limits_D k(x_1-\xi,\ y_1-\eta,\ z_1)\mathbf{q}_0(\xi,\ \eta)d\xi\delta\eta \tag{2.1}$$

$$k(\mathbf{x}_1)=\mathcal{F}^{-1}[K]=\frac{1}{4\pi^2}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2} K(\alpha_1,\ \alpha_2,\ z_1)e^{-i(\alpha_1 x_1+\alpha_2 y_1)}d\alpha_1 d\alpha_2 \tag{2.2}$$

Here $\mathcal{F}^{-1}$ is the inverse double Fourier transformation operator. The contours of integration Γ_1 and Γ_2 are situated in the complex plane α_1 and α_2 along the real axes, deviating from them only when circumventing real poles and branching points of the integrand. The direction of circumvention is dictated by the limiting absorption principle [6]. The explicit form of the elements of the matrix K for a uniform half-space is given, for example, in [6, 8]. For further discussion it is important that the relation between K and z_1, should have the form

$$K(\alpha_1,\ \alpha_2,\ z_1)=\sum_{n=1}^{2} K_n(\alpha_1,\ \alpha_2)e^{\sigma_n z_1},\quad z_1\leqslant 0$$

$$\sigma_n=\sqrt{\alpha^2-\varkappa_n^2},\quad \operatorname{Re}\sigma_n\geqslant 0,\quad \operatorname{Im}\sigma_n\leqslant 0$$

$$\alpha^2=\alpha_1^2+\alpha_2^2;\quad \varkappa_1=\omega/v_p,\quad \varkappa_2=\omega/v_s$$

(v_p and v_s are the velocities of the P- and S-waves).

Representation (2.1) gives the exact wave pattern over the whole half-space, but here we have to evaluate multiple integrals. For deep cracks, situated at a distance greater than several wavelengths λ from the surface ($|z_1|\gg\lambda$), it is quite sufficient to use their asymptotic representation. In the case in question the stationary points

$$\alpha_{1,n}=-\varkappa_n\cos\varphi\sin\psi,\quad \alpha_{2,n}=-\varkappa_n\sin\varphi\sin\psi$$

of the exponential components $\exp(\sigma_n z_1-i(\alpha_1 x_1+\alpha_2 y_1))$ of the integrands make the main contribution to the asymptotic form of integral (2.2) [8]. The corresponding asymptotic representation for the matrix k in (2.1) takes the form

$$k(x_1-\xi,\ y_1-\eta,\ z_1)=\sum_{n=1}^{2} k_n(\varphi,\ \psi)\frac{e^{i\varkappa_n R}}{R}+O(R^{-2}) \tag{2.3}$$

$$k_n(\varphi,\ \psi)=i\cos\psi\varkappa_n K_n(\alpha_{1,n},\ \alpha_{2,n})/(2\pi)$$

$$R=\sqrt{(x_1-\xi)^2+(y_1-\eta)^2+z_1^2}\to\infty,\quad \psi>\pi/2$$

Here φ and ψ are the angles of a spherical system of coordinates with centre at the current integration point (ξ, η, 0).

Replacing integral (2.1) by the approximate cubature formula in the nodes (ξ_m, η_m) (m = 1, 2, ..., M) and taking (2.3) into account we obtain the representation

$$\mathbf{u}_0^{\text{inc}}(\mathbf{x}_1)=\sum_{n=1}^{2}\mathbf{u}_{0n}(\mathbf{x}_1)\approx\sum_{n=1}^{2}\sum_{m=1}^{M}\mathbf{a}_{nm}\frac{e^{i\varkappa_n R_m}}{R_m} \tag{2.4}$$

$$\mathbf{a}_{nm}=k_n(\varphi_m,\psi_m)\mathbf{q}_0(\xi_m,\eta_m)s_m$$

in which φ_m, ψ_m, R_m are the coordinates of the point $\mathbf{x}_1$ considered in spherical systems connected with the nodes ξ_m, η_m, and s_m are weighting factors of the cubature formula (for a uniform grid with step h we have $s_m = h^2$).

The vector functions $\mathbf{u}_{01}$ and $\mathbf{u}_{02}$ describe volume P- and S-waves, respectively, excited in the far zone ($|z_1|$, $R_m\gg\lambda$) of the surface load $\mathbf{q}_0$. According to representation (2.4) they are approximated by the superposition of spherical waves travelling from elementary sources specified at the nodes of the grid (ξ_m, η_m).

From the specified $\mathbf{u}_0=\mathbf{u}_0^{\text{inc}}$ we calculate the stresses $\tau_0=T_z\mathbf{u}_0$, required to form the right-hand side of integral equation (1.4), where for $\mathbf{x}_1\in\Omega$ (in the zone far from the source) the action of the derivatives occurring in T_z on the components $\mathbf{u}_0^{\text{inc}}$ (2.4), apart from terms $O(R_m^{-2})$, reduces to multiplication by certain functions of the spherical angles [8]

$$\frac{\partial}{\partial x_1} \to i\varkappa_n \cos\varphi_m \sin\psi_m, \quad \frac{\partial}{\partial y_1} \to i\varkappa_n \sin\varphi_m \sin\psi_m, \quad \frac{\partial}{\partial z_1} \to i\varkappa_n \cos\psi_m$$

Although, for representation (2.4) to be applicable, the dimensions of the cells of the grid must be much less than the wavelength, this limitation does not lead to any appreciable costs in computing the field $\mathbf{u}_0$, since, in practice, the dimensions of the transducer are usually less than or comparable with the wavelength of the excited waves. In some special cases, for example, for a load $\mathbf{q}_0$ uniformly distributed over a circular region, asymptotic representation (2.3) can be integrated with respect to ξ and η in explicit form [9], which eliminates the need to use the approximation with respect to the angles (ξ_m, η_m).

The distribution of the contact stresses $\mathbf{q}_0$ depends on the type of source considered. For a source of vertical-type P-waves $\mathbf{q}_0 = \{0, 0, \sigma_z\}$ and for shear waves $\mathbf{q}_0 = \{\tau_{xz}, 0, 0\}$ (here σ_z and τ_{xz} are constants or are defined in terms of the solution of the corresponding contact problem).

In general, the so-called directional transducer of an electric signal excites a plane P- or S-wave in the piezoelectric crystal, which is incident at a certain angle γ on its contact zone with the surface of the material (the upper right corner of Fig. 1). Refracted and partially reflected at the interface of the media, this wave also excites a field $\mathbf{u}_0^{\text{inc}}$ in the material being probed. It is quite sufficient to take as $\mathbf{q}_0$ the stress distribution obtained when a plane wave is incident on the interface of two elastic media [3]. The solution of this problem is well known [10], where here it is also easy to take into account the specific contact conditions of the transducer, which may be either rigidly glued to the surface, or transmitting transverse oscillations (for numerical examples see the reference cited in the footnote).

In the case of interfacial cracks, when the field $\mathbf{u}_0^{\text{inc}}$ interacts with the interface $z_1 = -h_1$ additional reflected, refracted and channelled (Stonely) waves arise, which must be taken into account in $\mathbf{u}_0$ and τ_0 when solving integral equation (1.4) and calculating $\delta\Gamma$. In this case, to construct the asymptotic form of $\mathbf{u}_0$, one can also use an integral representation of the form (2.1), but with a kernel k for a double-layer half-space. In the asymptotic form obtained in this case we will take into account not only the boundary $z_1 = -h_1$ but also second-time reflection from the surface $z_1 = 0$. The latter is necessary when investigating diffraction by surface cracks, when the value of h is comparatively small, for example, when monitoring the peeling of thin films. However, for the deep-lying cracks being considered here ($h \gg \lambda$) the contribution of second-time reflections from $z_1 = 0$ can be neglected, thus reducing the asymptotic form $\mathbf{u}_0$ to

$$\mathbf{u}_0 \sim \begin{cases} \mathbf{u}_0^{\text{inc}} + \mathbf{u}_0^+, & z_1 \geqslant -h \\ \mathbf{u}_0^-, & z_1 \leqslant -h \end{cases} \quad \text{as } |z_1| = |z - h| \to \infty \tag{2.5}$$

where $\mathbf{u}_0^{\mp}$ are the waves reflected and refracted at the boundary $z_1 = -h$ ($z = 0$ in the crack system of coordinates), considered as wave fields in the upper ($z \geqslant 0$) and lower ($z \leqslant 0$) half-spaces, excited by certain loads $\mathbf{q}^{\pm}$, applied to the surface $z = 0$. Like $\mathbf{u}_0^{\text{inc}}$, these fields can be specified exactly in integral form (2.1) using Green's matrices $k^{\pm}$ of the corresponding half-spaces.

The coordinate system $\mathbf{x}$ of the crack in this case is related to the system $\mathbf{x}_1$ by the equation $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_c$, where $\mathbf{x}_c = \{x_c, y_c, -h\}$ is the coordinate of the centre of the crack in system $\mathbf{x}_1$. A Fourier transformation of integral representations of the form (2.1) with respect to x, y for $\mathbf{u}_0^{\text{inc}}$ and $\mathbf{u}_0^{\mp}$ leads to the following functional matrix relations

$$\begin{gathered} \mathbf{U}_0^{\text{inc}}(\alpha_1, \alpha_2, z) = K(\alpha_1, \alpha_2, z - h)\mathbf{Q}_0(\alpha_1, \alpha_2)e_0 \\ \mathbf{U}_0^{\pm}(\alpha_1, \alpha_2, z) = K^{\pm}(\alpha_1, \alpha_2, z)\mathbf{Q}^{\pm}(\alpha_1, \alpha_2) \\ e_0 = \exp[-i(\alpha_1 x_c + \alpha_2 y_c)] \end{gathered} \tag{2.6}$$

The symbols K and K^- of Green's matrices for the lower half-spaces $z_1 \leqslant 0$ and $z \leqslant 0$ have the same form but differ in the values of the constants occurring in them (the wave numbers $\varkappa_1$ and $\varkappa_2$ for the velocities v_p and v_s) for the upper and lower materials ($-h \leqslant z_1 \leqslant 0$) and ($z_1 \leqslant -h$) respectively. The form of K^+ for the upper half-space is obtained from K^- by replacing σ_n by $-\sigma_n$ ($n = 1, 2$) and also differs in the values of the material constants.

The sewing conditions at the boundary $z = 0$

$$\mathbf{u}_0^{\text{inc}} + \mathbf{u}_0^+ = \mathbf{u}_0^-, \quad \boldsymbol{\tau}_0^{\text{inc}} + \mathbf{q}_0^+ = \mathbf{q}_0^- \tag{2.7}$$

enable us to eliminate the unknowns $\mathbf{Q}^{\pm}$ by expressing $\varkappa$ also in terms of $\mathbf{Q}_0$ (the arguments α_1 and α_2 – the Fourier transformation parameters – are largely omitted here and henceforth)

$$\mathbf{Q}^- = L(K_0^+ T_h - K_h)\mathbf{Q}_0 e_0, \quad \mathbf{Q}^+ = \mathbf{Q}^- - T_h \mathbf{Q}_0 e_0 \tag{2.8}$$

$$L = [K_0^+ - K_0^-]^{-1}, \quad K_0^{\pm} = K^{\pm}(0)$$

$$K_h = K(-h) = \sum_{n=1}^{2} K_n e^{-\sigma_n h}, \quad T_h = T_z K(z)|_{z=-h} = \sum_{n=1}^{2} T_n K_n e^{-\sigma_n h}$$

$T_n = T_z(\alpha_1, \alpha_2, \sigma_n)$, $T_z(\alpha_1, \alpha_2, d/dz)$ is the matrix Fourier transformation operator of the stresses for the area with normal Oz.

Correspondingly, from relations (2.5) and (2.6) when $z = 0$

$$\mathbf{U}_0(0) \approx K_0^- \mathbf{Q}^- = \sum_{n=1}^{2} K_{0n} e^{-\sigma_n h} e_0 \mathbf{Q}_0 \tag{2.9}$$

with $K_{0n} = K_0^- L(K_0^+ T_n - E)K_n$ and for the field $\mathbf{u}_0(x, y, 0) = \mathcal{F}^{-1}[\mathbf{U}_0(0)]$ incident on the crack, we arrive at an asymptotic representation of the same form as (2.4) in which

$$R_m = [(x + x_c - \xi_m)^2 + (y + y_c - \eta_m)^2 + h^2]^{1/2}$$

φ_m and ψ_m are the corresponding angular coordinates of the point $\mathbf{x} \in \Omega$ in a spherical system with centre $(\xi_m, \eta_m, 0)$, and the matrices K_n are replaced by K_{0n}.

The asymptotic representation obtained in this way takes into account both the initial wave field of the source $\mathbf{u}_0^{\text{inc}}$ and the additional volume waves which arise when it interacts with the boundary between the materials. As regards the excited Stonely-type channelled waves, they ae described by the contribution of the residues to the real poles U_0 close to the real axis. In the case in question they are specified by poles of the elements of the matrix L in (2.8) (see relations (3.1) below), i.e. by the zeros of the determinant of the matrix $K_0^+ - K_0^-$, which are identical with the characteristic equation for the Stonely wave. Channelled waves play an important role in remote sounding along the connection boundaries, for example, in the flaw detection of the welded joints of pipelines. Their contribution to the overall asymptotic form of the incident field $\mathbf{u}_0$ was also taken into account in this model.

3. THE INTEGRAL EQUATION

By construction [4], the matrix L (α_1, α_2), occurring in expression (2.8) is the symbol of the kernel of integral equation (1.4). For the joining of materials of different moduluses

$$L(\alpha_1, \alpha_2, \alpha) = \begin{Vmatrix} (i\alpha_2^2 N_0 - \alpha_1^2 M_{01})/\alpha^2 & -\alpha_1\alpha_2(M_{01} + iN_0)/\alpha^2 & -i\alpha_1 P_0 \\ -\alpha_1\alpha_2(M_{01} + iN_0)/\alpha^2 & (i\alpha_1^2 N_0 - \alpha_2^2 M_{01})/\alpha^2 & -i\alpha_2 P_0 \\ i\alpha_1 P_0 & i\alpha_2 P_0 & -M_{02} \end{Vmatrix}$$

where

$$M_{0j} = \frac{M_j}{M_1 M_2 - \alpha^2 (P_2 - P_1)^2}, \quad j = 1, 2 \tag{3.1}$$

$$P_0 = \frac{P_2 - P_1}{M_1 M_2 - \alpha^2 (P_2 - P_1)^2}, \quad N_0 = \frac{i}{N_1 + N_2}$$

the functions M_j and N_j ($j = I_2$ 2) take the form

$$M_j(\alpha) = \frac{1}{2}\left(\varkappa_{21}^2 \frac{\sigma_{j1}}{\Delta_j} + \varkappa_{22}^2 \frac{\sigma_{j2}}{\Delta_j}\right), \quad N_j(\alpha) = \frac{1}{\mu_j \sigma_{2j}}$$

and

$$P_j(\alpha)=\frac{1}{\Delta_j}\left(\alpha^2-\frac{1}{2}\varkappa_{2j}^2-\sigma_{1j}\sigma_{2j}\right)$$

$$\Delta_j(\alpha)=2\mu_j\left(-\left(\alpha^2-\frac{1}{2}\varkappa_{2j}^2\right)^2+\alpha^2\sigma_{1j}\sigma_{2j}\right)$$

$$\sigma_{kj}=\sqrt{\alpha^2-\varkappa_{kj}^2},\ \operatorname{Re}\sigma_{kj}\geqslant 0,\ \operatorname{Im}\sigma_{kj}\leqslant 0,\ k,j=1,2$$

$$\varkappa_{1j}=\omega/v_{pj},\ \varkappa_{2j}=\omega/v_{sj},\ \alpha^2=\alpha_1^2+\alpha_2^2$$

Unlike the case of a crack in a uniform medium, the matrix L obtained here is filled and system (1.4) does not split into independent equations in the normal and tangential components of the displacement jump. Nevertheless the variational-difference method is completely applicable in this case. As a result of applying it, a program was written which enables approximate solutions of system (1.4) to be obtained on a personal computer. These solutions are necessary both to determine the scattering patterns and the energy scattering coefficient Σ when a plane wave is incident on an interfacial crack at an arbitrary angle, and also the coefficient $\delta\Gamma$ for the specified source. This enables one to take into account the influence of the ratios of the elastic properties of the materials, in addition to the shape and orientation of crack. For example, Fig. 3 illustrates the influence of the ratio of the velocities of the S-waves on the form of the dependence of Σ on the dimensionless parameter $\varkappa_{12}a$ for a square crack of side $2a$ for normal incidence of a plane P-wave.

It should be noted that, instead of (2.7) one can also consider other material joining conditions, for example, which describe non-absolute contact, when the deformation and elastic properties of a layer of infinitesimal thickness in a glued or welded connection is taken into account. These conditions are given by the matrix relation

$$\left\|\begin{matrix}\mathbf{u}^+\\ \boldsymbol{\tau}^+\end{matrix}\right\|=B\left\|\begin{matrix}\mathbf{u}^-\\ \boldsymbol{\tau}^-\end{matrix}\right\| \text{ when } z=0$$

in which the elements of the matrix B depend on the properties of the layer and the frequency [11], or in the form of a spring contact [12]

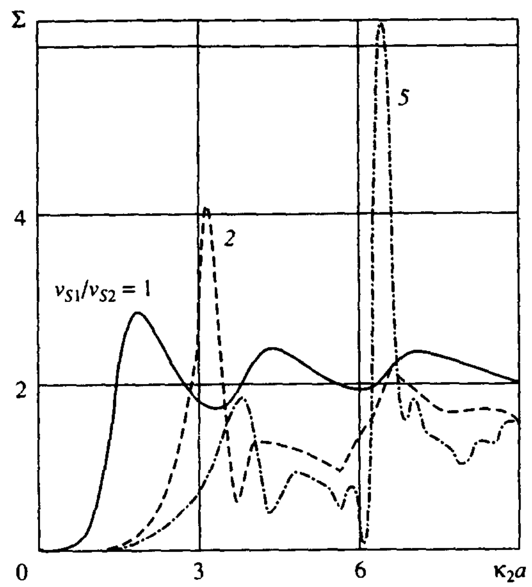

Fig. 3

$$\boldsymbol{\tau}^{+} = \boldsymbol{\tau}^{-} = \boldsymbol{\tau}, \ \mathbf{u}^{+} - \mathbf{u}^{+} = C\boldsymbol{\tau} \text{ when } z = 0 \tag{3.2}$$

The scheme described above for constructing a solution, on the whole, remains as before and only the specific form of the symbol of the kernel of the integral equation and the asymptotic forms of the wave fields are changes.

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